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Weak Disorder Expansion of Lyapunov Exponents of Products of Random Matrices: A Degenerate Theory

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The weak disorder expansion of Lyapunov exponents of products of random matrices is derived by a new method. Our treatment can be easily generalized to the problem when in the limit of zero randomness two eigenvalues of the matrices are equal. For real degenerate matrices, the formula for the leading term of the Lyapunov exponent is derived. It has the form of a continuous fraction, which converges quickly to the exact value.

KEY WORDS: Product of random matrices; Lyapunov exponents; degenerate perturbation.

1. INTRODUCTION

The problem of calculating the Lyapunov exponents (LE) of products of random matrices arises in the theory of disordered systems,[1,2] probability theory,[3-6] and dynamical systems.[7-9] While the methods for a numerical calculation of the LE have been described in detail, e.g., in studies of localization in disordered electronic systems,[10-12] an analytical treatment is successful only in a few special cases.[13] It is therefore useful to study the weak disorder expansion (WDE) of the LE.[14] Given a product of random $M \times M$ matrices,

$$\mathbf{X}^{(N)} = \mathbf{T}^{(N)}\mathbf{T}^{(N-1)} \cdots \mathbf{T}^{(2)}\mathbf{T}^{(1)}, \qquad \mathbf{X}^{(0)} = \mathbf{1} \tag{1}$$

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it is possible, following ref. 14, to construct the WDE of all LE. In (1), the matrices $\mathbf{T}^{(l)}$ have the form

$$\mathbf{T}^{(l)} = \mathbf{A} + \mu \mathbf{B}^{(l)} \tag{2}$$

where $\mathbf{A}$ is supposed to be diagonal with eigenvalues $\lambda_1, \lambda_2,..., \lambda_M$, and the statistically independent random matrices $\mathbf{B}^{(l)}$ represent the randomness. We suppose that the probability distributions of the matrix elements $B_{ij}^{(l)}$ do not depend on l and that for all $l \neq k$

$$\langle \mathbf{B}^{(l)} \rangle = 0, \qquad \langle \mathbf{B}^{(l)} \mathbf{B}^{(k)} \rangle = 0 \tag{3}$$

The parameter μ measures the strength of the randomness.

The construction of the WDE may, however, meet serious difficulties. Namely, it works only under the condition that all eigenvalues of the matrix $\mathbf{A}$ differ in the absolute values:

$$|\lambda_i| > |\lambda_j|, \qquad i < j \tag{4}$$

If the matrix $\mathbf{A}$ has a degenerate spectrum, i.e.,

$$\lambda_i = \lambda_j, \qquad i \neq j \tag{5}$$

then an another treatment is necessary. This problem has been solved for real matrices in ref. 15.

In this paper, we propose a new method of construction of the WDE of LE for the nondegenerate case. We develop the new formalism, which enables us to treat the sum of the p largest LE of product (1) as the first (largest) LE of the infinite product of other random matrices $\mathbf{T}^{[p](l)}$. Our formulas lead to the well-known results as presented in ref. 14 (Section 2). They also show the origin of "anomalies" of the WDE which appears under the condition (5).

The most important part of this paper is Section 3. Here, we derive the *degenerate* perturbation theory for *real* 2×2 matrices T. Our results hold, after appropriate redefinition of the matrices $\mathbf{T}$, also for the $M \times M$ matrices with two degenerate eigenvalues. We obtain the leading term of the larger LE γ_1 exactly as a continuous fraction, which for all examples we have studied converges very well. As follows from our considerations, this treatment is also applicable for complex matrices of the form

$$\mathbf{T} = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \tag{6}$$

We study in this section also the WDE of LE of real matrices, the eigenvalues of which satisfy the relation

$$\frac{\lambda_i}{\lambda_{i+1}} = e^{-2i\varphi}, \qquad \varphi \text{ real} \tag{7}$$

and show that the "anomaly" in WDE appears only if $\varphi = \pi r/s$ with r, s integers. In Section 4 we compare our treatment with another method published previously.[15,20]

2. THE NONDEGENERATE THEORY

2.1. The Starting Formulas

If the matrix $\mathbf{A}$ has a nondegenerate spectrum, i.e., if (4) holds, then the *largest* LE γ_1 can be defined in the standard way[14]:

$$\gamma_1 = \lim_{N\to\infty} \frac{1}{N} \log \frac{X_{11}^{(N)}}{X_{11}^{(0)}} = \lim_{N\to\infty} \frac{1}{N} \sum_{l=1}^{N} \log \frac{X_{11}^{(l)}}{X_{11}^{(l-1)}} \tag{8}$$

where, owing to (1),

$$\mathbf{X}^{(l)} = \mathbf{T}^{(l)}\mathbf{X}^{(l-1)} \tag{9}$$

A similar formula for the sum

$$\Gamma_p = \gamma_1 + \gamma_2 + \cdots + \gamma_p \tag{10}$$

of the p largest LE can be derived as follows: Let us introduce the ordered sets of p integers

$$\omega_\alpha = (i_1, i_2,..., i_p), \qquad 1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant M \tag{11}$$

There are $M_p = \binom{M}{p}$ such different sets. Having two different sets $\omega_\alpha = (i_1, i_2,..., i_p)$ and $\omega_\beta = (j_1, j_2,..., j_p)$, we define $\alpha < \beta$ if the condition $i_k \leqslant j_k$ holds for all $k = 1, 2,..., p$. Evidently, $\omega_1 = (1, 2,..., p)$, $\omega_2 = (1, 2,..., p-1, p+1)$, and $\omega_{M_p} = (M+1-p,..., M)$.

Let us now define the new $M_p \times M_p$ matrices $\mathbf{T}^{[p](l)}$ ($\mathbf{X}^{[p](l)}$), the elements of which, $T_{\alpha\beta}^{[p](l)}$ ($X_{\alpha\beta}^{[p](l)}$), are equal to subdeterminants of the matrix $\mathbf{T}^{(l)}$ ($\mathbf{X}^{(l)}$), constructed from its p rows, labeled according to array ω_α, and p columns, labeled according to array ω_β. As is shown in Appendix A, these new matrices satisfy the same recursive relation as the original ones:

$$\mathbf{X}^{[p](l+1)} = \mathbf{T}^{[p](l+1)}\mathbf{X}^{[p](l)}, \qquad l = 1, 2,... \tag{12}$$

Γ_p is therefore nothing else but the largest LE of the product of random matrices $\prod_l^N \mathbf{T}^{[p](l)}$. As, owing to (4), the largest eigenvalue

$$\lambda_1^{[p]} = \lambda_1 \lambda_2 \cdots \lambda_p \tag{13}$$

of $\mathbf{A}^{[p]}$ is nondegenerate, we can define Γ_p in the same way as γ_1, but with matrices $\mathbf{X}$ replaced by $\mathbf{X}^{[p]}$:

$$\Gamma_p = \lim_{N\to\infty} \frac{1}{N} \log \frac{X_{11}^{[p](N)}}{X_{11}^{[p](0)}} = \lim_{N\to\infty} \frac{1}{N} \sum_{l=1}^{N} \log \frac{X_{11}^{[p](l)}}{X_{11}^{[p](l-1)}} \tag{14}$$

Note that this definition coincides with Eq. (9) of ref. 14.

2.2. The Weak Disorder Expansion

Using the recurrence relation (12), one can rewrite (14) as

$$\Gamma_p = \lim_{N\to\infty} \frac{1}{N} \sum_{l=1}^{N} \log \left\{ T_{11}^{[p](l)} + \sum_{\alpha\neq 1}^{M_p} T_{1\alpha}^{[p](l)} z_\alpha^{[p](l-1)} \right\} \tag{15}$$

In (15) we have introduced new variables $z_\alpha^{[p](l)}$,

$$z_\alpha^{[p](l)} = \frac{X_{\alpha 1}^{[p](l)}}{X_{11}^{[p](l)}} \tag{16}$$

which satisfy the recursive relation

$$z_\alpha^{[p](l)} = \frac{T_{\alpha 1}^{[p](l)} + \sum_{\beta\neq 1}^{M_p} T_{\alpha\beta}^{[p](l)} z_\beta^{[p](l-1)}}{T_{11}^{[p](l)} + \sum_{\beta\neq 1}^{M_p} T_{1\beta}^{[p](l)} z_\beta^{[p](l-1)}} \tag{17}$$

To construct the WDE of Γ_p, we extract all terms of order 1 in (15). We define therefore the new matrix $t_{\alpha\beta}^{[p](l)} \sim \mu$ as

$$T_{\alpha\beta}^{[p](l)} = \lambda_1^{[p]} (\kappa_\alpha^{[p]} \delta_{\alpha\beta} + t_{11}^{[p](l)}), \qquad \kappa_\alpha^{[p]} = \frac{\lambda_\alpha^{[p]}}{\lambda_1^{[p]}} \tag{18}$$

and rewrite (15) as

$$\Gamma_p = \log \lambda_1^{[p]} + \lim_{N\to\infty} \frac{1}{N} \sum_{l=1}^{N} \log \left\{ 1 + t_{11}^{[p](l)} + \sum_{\alpha\neq 1}^{M_p} t_{1\alpha}^{[p](l)} z_\alpha^{[p](l-1)} \right\} \tag{19}$$

The quantities $z_\alpha^{[p](l)}$ then satisfy the recursive relation

$$z_\alpha^{[p](l)} = \frac{\kappa_\alpha^{[p]} z_\alpha^{[p](l-1)} + t_{\alpha 1}^{[p](l)} + \sum_{\beta\neq 1} t_{\alpha\beta}^{[p](l)} z_\beta^{[p](l-1)}}{1 + t_{11}^{[p](l)} + \sum_{\beta\neq 1} t_{1\beta}^{[p](l)} z_\beta^{[p](l-1)}} \tag{20}$$

Using (19) and (20), we can obtain the WDE of Γ_p as follows:

$$\begin{aligned}\Gamma_p \approx{} & \log \lambda_1^{[p]} + \langle t_{11}^{[p]} \rangle - \tfrac{1}{2}\langle t_{11}^{[p]2} \rangle + \tfrac{1}{3}\langle t_{11}^{[p]3} \rangle - \tfrac{1}{4}\langle t_{11}^{[p]4} \rangle \\ & + \sum_{\alpha \neq 1}^{M_p} (\langle t_{1\alpha}^{[p]} \rangle - \langle t_{11}^{[p]} t_{1\alpha}^{[p]} \rangle)\, \overline{z_\alpha^{[p]}} \\ & - \tfrac{1}{2} \sum_{\alpha, \beta \neq 1}^{M_p} \langle t_{1\alpha}^{[p]} t_{1\beta}^{[p]} \rangle\, \overline{z_\alpha^{[p]} z_\beta^{[p]}} + \cdots \end{aligned} \tag{21}$$

where we have supposed that the quantities

$$\begin{aligned} \overline{z_\alpha^{[p]}} &= \lim_{N \to \infty} \frac{1}{N} \sum_{l=1}^{N} z_\alpha^{[p](l)} \\ \overline{z_\alpha^{[p]} z_\beta^{[p]}} &= \lim_{N \to \infty} \frac{1}{N} \sum_{l=1}^{N} z_\alpha^{[p](l)} z_\beta^{[p](l)} \\ &\vdots \end{aligned} \tag{22}$$

exist. To obtain them, we expand the right-hand side of (20) into a power series in μ,

$$\begin{aligned} z_\alpha^{[p](l)} \sim{} & t_{\alpha 1}^{[p](l)} - t_{\alpha 1}^{[p](l)} t_{11}^{[p](l)} + \cdots \\ & + \sum_{\beta \neq 1}^{M_p} z_\beta^{[p](l-1)} \{ (t_{\alpha\beta}^{[p](l)} + \kappa_\alpha^{[p]} \delta_{\alpha\beta})(1 - t_{11}^{[p](l)}) - t_{\alpha 1}^{[p](l)} t_{1\beta}^{[p](l)} + \cdots \} \\ & + \sum_{\beta\delta \neq 1}^{M_p} z_\beta^{[p](l-1)} z_\delta^{[p](l-1)} \{ -(t_{\alpha\beta}^{[p](l)} + \kappa_\alpha^{[p]} \delta_{\alpha\beta})\, t_{1\delta}^{[p](l)} + \cdots \} \\ & + \cdots \end{aligned} \tag{23}$$

where dots stand for higher-order terms. After averaging (23) over all t's, using the definition of matrices T, and over z's using (22), we obtain the linear relations for the quantities

$$\overline{z_\alpha^{[p]}},\ \overline{z_\alpha^{[p]} z_\beta^{[p]}}, \ldots$$

In Appendix B we present their WDE up to the second order in μ. Using formulas (B1)–(B9), one can easily recover the WDE presented previously.(14,15)

3. DEGENERATE PERTURBATION THEORY

3.1. General Considerations

Suppose now that an $M \times M$ matrix A has a degenerate spectrum, i.e., that $\lambda_p = \lambda_{p+1}$. Following the considerations from the previous section, we can construct matrices $\mathbf{T}^{[p]}$, $\mathbf{X}^{[p]}$; the first two eigenvalues of the matrix $\mathbf{A}^{[p]}$ are now degenerated:

$$\lambda_1^{[p]}=\lambda_1\lambda_2\cdots\lambda_p=\lambda_2^{[p]}=\lambda_1\cdots\lambda_{p-1}\lambda_{p+1}>\lambda_\beta^{[p]},\qquad \beta>2$$

On the basis of considerations from the previous section and of the formulas from Appendix B, we easily verify that $\overline{z_\beta}\sim\mu^2$ for all $\beta>2$. Therefore, to obtain the second-order term of the WDE of Γ_p, it is enough to consider only the 2×2 matrices

$$\begin{pmatrix} T_{11}^{[p](l)} & T_{12}^{[p](l)} \\ T_{21}^{[p](l)} & T_{22}^{[p](l)} \end{pmatrix}$$

That is why in what follows we can restrict ourselves to the problem of calculating the *larger* LE γ_1 of the products of 2×2 matrices

$$\mathbf{T}^{(l)}=\begin{pmatrix} T_{11}^{(l)} & T_{12}^{(l)} \\ T_{21}^{(l)} & T_{22}^{(l)} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\begin{pmatrix} t_{11}^{(l)} & t_{12}^{(l)} \\ t_{21}^{(l)} & t_{22}^{(l)} \end{pmatrix} \tag{24}$$

The second LE, γ_2, is easy to obtain, using formulas from the previous section (see also ref. 15), as

$$\gamma_2=\langle\log\det\mathbf{T}\rangle-\gamma_1 \tag{25}$$

In (24), the elements t_{ij} are given by formulas (B1)–(B3). One sees from (B1)–(B3) that the matrix elements t_{ij} in (24) also contain in general terms $\sim\mu^2$. To simplify the notation, we have not extracted these terms from t. Thus, instead of the assumption $\langle t_{ij}\rangle=0$, we have in general to consider $\langle t_{ij}\rangle\sim\mu^2$.

We consider throughout this section all elements t_{ij} *real*. Then we are concerned in what follows only with matrices for which

$$\langle t_{12}^2\rangle\neq 0 \qquad \text{and} \qquad \langle t_{21}^2\rangle\neq 0 \tag{26}$$

because the opposite assumption implies for any l either $t_{12}^{(l)}\equiv 0$ or $t_{21}^{(l)}\equiv 0$, respectively. For such a trivial case, however, no degenerate theory is necessary, because evidently $\gamma_1=-\min\{\langle t_{11}^2\rangle, \langle t_{22}^2\rangle\}/2+O(\mu^4)$.

Before proceeding further, it is necessary to introduce some general remarks. Let us go back to the formulas (21)–(23). To derive them, we have *a priori* supposed that both the logarithm in (19) and the fraction in (20) can be expanded in powers of μ and z. Such an assumption, however, requires that the condition

$$\mu^2 z_\alpha^{(l)}\ll 1 \qquad \text{for all} \quad \alpha, l \tag{27}$$

holds. The validity of the inequality (27) is closely connected to the nondegeneracy of eigenvalues of the matrix $\mathbf{A}$. It is namely equivalent to the assumption that the scalar product of the vector $\mathbf{y}^{(l)}$,

$$\mathbf{y}^{(l)} = \begin{pmatrix} X_{11}^{(l)} \\ X_{21}^{(l)} \\ \vdots \\ X_{M_p 1}^{(l)} \end{pmatrix} = \mathbf{T}^{(l)} \cdots \mathbf{T}^{(1)} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{28}$$

with the "original" vector $(1, 0, \ldots)$ is completely determined by its first component $X_{11}^{(l)}$ for *all* l. This assumption, however, is correct only if the difference $\lambda_1^{[p]} - \lambda_\alpha^{[p]}$ is large enough. We estimate the last condition as

$$\left| \frac{\lambda_2^{[p]}}{\lambda_1^{[p]}} \right| \ll 1 - \mu^2 \tag{29}$$

If (29) holds, one can use the formulas given in the previous section and one obtains the second-order term of the WDE of LE as a power series in Δ^{-1}, $\Delta = (\lambda_1 - \lambda_2)/2\mu^2$.[14]

For degenerate matrices, condition (29) is, of course, not valid. It is therefore not assured that condition (27) holds for each l. Then, however, the existence of the limits (22) is no longer guaranteed. Moreover, it is not possible to expand expressions (19), (20). Therefore, to construct the WDE, we need an another starting formula. The natural generalization of the nondegenerate treatment is

$$\gamma_1 = \lim_{N \to \infty} \frac{1}{N} \sum_{l=1}^{N} \log \frac{y_1^{(l)} + i y_2^{(l)}}{y_1^{(l-1)} + i y_2^{(l-1)}} \tag{30}$$

with variables y_1, y_2 satisfying the recursive relations

$$\begin{aligned} y_1^{(l)} &= T_{11}^{(l)} y_1^{(l-1)} + T_{12}^{(l)} y_2^{(l-1)} \\ y_2^{(l)} &= T_{21}^{(l)} y_1^{(l-1)} + T_{22}^{(l)} y_2^{(l-1)} \end{aligned} \tag{31}$$

The real part of (30),

$$\operatorname{Re} \gamma_1 = \lim_{N \to \infty} \frac{1}{2N} \sum_{l=1}^{N} \log \frac{[y_1^{(l)}]^2 + [y_2^{(l)}]^2}{[y_1^{(l-1)}]^2 + [y_2^{(l-1)}]^2} \tag{32}$$

determines the growth of the length of the vector $\mathbf{y}^{(l)} = (y_1^{(l)}, y_2^{(l)})$ and coincides with the definition proposed, for instance, in ref. 15. The imaginary part,

$$\begin{aligned}\operatorname{Im}\gamma_1 &= \lim_{N\to\infty}\frac{1}{N}\sum_{l=1}^{N}\arctan\frac{y_2^{(l)}y_1^{(l-1)}-y_1^{(l)}y_2^{(l-1)}}{y_1^{(l)}y_1^{(l-1)}+y_2^{(l)}y_2^{(l-1)}}\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{l=1}^{N}\arctan\frac{y_2^{(l)}}{y_1^{(l)}}-\arctan\frac{y_2^{(l-1)}}{y_1^{(l-1)}}\end{aligned} \tag{33}$$

describes the rotation of the starting vector $\mathbf{y}^{(0)}$ in the (y_1, y_2) plane, and also could be of interest.

3.2. The Construction of the Degenerate Expansion

To construct the WDE of γ_1, let us rewrite (30) in the form

$$\gamma_1 = \lim_{N\to\infty}\frac{1}{N}\sum_{l=1}^{N}\log\{1+\tau_{11}^{(l)}+\tau_{12}^{(l)}u^{(l-1)}\} \tag{34}$$

with matrix τ defined as

$$\tau = \frac{1}{2}\begin{pmatrix} t_{11}+t_{22}+i(t_{21}-t_{12}) & t_{11}-t_{22}+i(t_{12}+t_{21}) \\ t_{11}-t_{22}-i(t_{12}+t_{21}) & t_{11}+t_{22}-i(t_{21}-t_{12}) \end{pmatrix} \tag{35}$$

In (34), we have introduced the new variable u as

$$u^{(l)} = \frac{y_1^{(l)}-iy_2^{(l)}}{y_1^{(l)}+iy_2^{(l)}} \tag{36}$$

for which the recurrence formula

$$u^{(l)} = \frac{u^{(l-1)}+\tau_{21}^{(l)}+\tau_{22}^{(l)}u^{(l-1)}}{1+\tau_{11}^{(l)}+\tau_{12}^{(l)}u^{(l-1)}} \tag{37}$$

similar to (20), holds. As

$$|u^{(l)}| = 1 \qquad \text{for all} \quad l \tag{38}$$

we see that the expansion of the right-hand side of (34), (37) is always possible. We can therefore proceed further in the same way as in Section 2: Supposing that variables u_n,

$$u_n = \lim_{N\to\infty}\frac{1}{N}\sum_{l=1}^{N}[u^{(l)}]^n, \qquad n = 1, 2,... \tag{39}$$

exist, we obtain for them from (37) the following system of equations:

$$u_n = u_n + \mu^2 f_n + O(\mu^4) \tag{40}$$

where

$$f_n = \mathscr{C}_{n,n-2}u_{n-2} + \mathscr{C}_{n,n-1}u_{n-1} + \mathscr{C}_{n,n}u_n + \mathscr{C}_{n,n+1}u_{n+1} + \mathscr{C}_{n,n+2}u_{n+2} \tag{41}$$

The coefficients $\mathscr{C}$ have the following form:

$$\begin{aligned}
\mathscr{C}_{n,n-2} &= \lim_{\mu\to 0} \frac{1}{\mu^2}\,[(n-1)\langle\tau_{21}^2\rangle] \\
\mathscr{C}_{n,n-1} &= \lim_{\mu\to 0} \frac{1}{\mu^2}\,\{-2n[\langle(\tau_{11}-\tau_{22})\,\tau_{21}\rangle] - 2\langle\tau_{21}\tau_{22}\rangle + 2\langle\tau_{21}\rangle\} \\
\mathscr{C}_{n,n} &= \lim_{\mu\to 0} \frac{1}{\mu^2}\,\{\langle\tau_{11}^2\rangle - \langle\tau_{22}^2\rangle + n[\langle(\tau_{11}-\tau_{22})^2\rangle - 2\langle\tau_{12}\tau_{21}\rangle] \\
&\quad + 2\langle\tau_{22}-\tau_{11}\rangle\} \\
\mathscr{C}_{n,n+1} &= \lim_{\mu\to 0} \frac{1}{\mu^2}\,\{2n[\langle(\tau_{11}-\tau_{22})\,\tau_{12}\rangle] + 2\langle\tau_{12}\tau_{11}\rangle - 2\langle\tau_{12}\rangle\} \\
\mathscr{C}_{n,n+2} &= \lim_{\mu\to 0} \frac{1}{\mu^2}\,[(n+1)\langle\tau_{12}^2\rangle]
\end{aligned} \tag{42}$$

Thus, the "absolute" terms in (40), $\sim\mu^0$, cancel each other. The existence of limits (39) requires then that $f_n = 0$, i.e.,

$$\mathscr{C}_{n,n-2}u_{n-2} + \mathscr{C}_{n,n-1}u_{n-1} + \mathscr{C}_{n,n}u_n + \mathscr{C}_{n,n+1}u_{n+1} + \mathscr{C}_{n,n+2}u_{n+2} = 0 \tag{43}$$

(for $n = 1$, 2, Eq. (43) also provides the absolute term on the rhs).

Due to (37), we have $u_n \sim O(1)$ for all n. This result explains the origin of "anomalies" of the expansion of the LE: to obtain the first term of the expansion, one also has to find quantities u_1, u_2.

The five-diagonal system (43) has in general four "basic solutions" which behave as $e^{\chi n}$ for large n. In Appendix C we prove that if (26) holds, then χ always has a nonzero real part. Owing to (38), we are interested only in the exponentially decreasing solutions. However, as explained in Appendix C, one cannot find them without additional information. This is only possible if system (43) reduces to a three-diagonal one, i.e., if *for all n* either

$$\mathscr{C}_{n,n-1} = \mathscr{C}_{n,n+1} = 0 \tag{44}$$

or

$$\mathscr{C}_{n,n-2} = \mathscr{C}_{n,n+2} = 0 \tag{45}$$

As $\mathscr{C}_{n,n-1}$, $\mathscr{C}_{n,n+1}$ contain terms $\sim n$ together with terms $\sim n^0$, conditions (44) cannot be satisfied generally. Nevertheless, in Appendix D we show an example for which these conditions are satisfied. On the other hand, condition (45) is easily to satisfy. To do so, we use the fact that all the LE are invariant with respect to the similarity transformation

$$\mathbf{T} \to \tilde{\mathbf{T}} = \mathbf{U}^{-1}\mathbf{T}\mathbf{U} \tag{46}$$

as can be easily verified from (1). We can therefore first transform our matrix $\mathbf{T}$ in such a form that the conditions

$$\langle \tilde{\tau}_{12}^2 \rangle = \langle \tilde{\tau}_{21}^2 \rangle = 0 \tag{47}$$

are satisfied. Then the system (43) transforms to the more suitable one

$$\tilde{\mathscr{C}}_{n,n-1}\tilde{u}_{n-1} + \tilde{\mathscr{C}}_{n,n}\tilde{u}_n + \tilde{\mathscr{C}}_{n,n+1}\tilde{u}_{n+1} = 0 \tag{48}$$

The fact that the system (48) has only one exponentially decreasing solution for $n \to \infty$ enables us to calculate from it the quantities we need.

It is always possible to find the matrix $\mathbf{U}$ which secures the validity of conditions (47). Indeed, let us note that

$$\tau = \mathbf{Q}^{-1}(\mathbf{T} - \mathbf{1})\mathbf{Q} \tag{49}$$

where

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \tag{50}$$

Therefore, conditions (47) are fulfilled if the matrix $\mathbf{U}$ has the form

$$\mathbf{U} = \mathbf{Q}\mathbf{U}_0\mathbf{Q}^{-1}, \qquad \mathbf{U}_0 = \begin{pmatrix} 1 & x \\ y & 1 \end{pmatrix} \tag{51}$$

where x, y satisfy the biquadratic equations

$$\lim_{\mu \to 0} \frac{1}{\mu^2} [\langle \tau_{21}^2 \rangle x^4 + 2\langle \tau_{21}(\tau_{22} - \tau_{11}) \rangle x^3 + \langle (\tau_{11} - \tau_{22})^2 - 2\tau_{12}\tau_{21} \rangle x^2$$
$$+ 2\langle \tau_{12}(\tau_{11} - \tau_{22}) \rangle x + \langle \tau_{12}^2 \rangle] = 0 \tag{52}$$

$$\lim_{\mu \to 0} \frac{1}{\mu^2} [\langle \tau_{21}^2 \rangle + 2\langle \tau_{21}(\tau_{22} - \tau_{11}) \rangle y + \langle (\tau_{11} - \tau_{22})^2 - 2\tau_{12}\tau_{21} \rangle y^2$$
$$+ 2\langle \tau_{12}(\tau_{11} - \tau_{22}) \rangle y^3 + \langle \tau_{12}^2 \rangle y^4] = 0 \tag{53}$$

As is shown in Appendix C, Eqs. (52) and (53) have no solution x_0 such that $|x_0| = 1$. We can therefore use the symmetry of the coefficients in (52) and (53) and take $y = x^*$. The matrix **U** then reads

$$\mathbf{U} = \frac{1}{2(1-|x|^2)^{1/2}} \begin{pmatrix} 2+(x+x^*) & i(x^*-x) \\ i(x^*-x) & 2-(x+x^*) \end{pmatrix} \tag{54}$$

i.e., it is real. Then $\tilde{\mathbf{T}}$ is real, too, and we can follow all our considerations from formulas (30) with the matrix $\tilde{\mathbf{T}}$ instead of **T**. Instead of (43), we obtain then the three-diagonal system (48), from which it is possible to obtain $\tilde{u}_1$ exactly as

$$\tilde{u}_1 = \cfrac{-\tilde{\mathscr{C}}_{1,0}}{\tilde{\mathscr{C}}_{1,1} - \cfrac{\tilde{\mathscr{C}}_{1,2}\tilde{\mathscr{C}}_{2,1}}{\tilde{\mathscr{C}}_{2,2} - \cfrac{\tilde{\mathscr{C}}_{2,3}\tilde{\mathscr{C}}_{3,2}}{\tilde{\mathscr{C}}_{3,3} - \cdots}}} \tag{55}$$

where the coefficients $\tilde{\mathscr{C}}_{nn}$, $\tilde{\mathscr{C}}_{n,n\pm 1}$ are defined through elements of the matrix

$$\tilde{\tau} = \mathbf{U}_0^{-1} \tau \mathbf{U}_0 \tag{56}$$

in the same way as in relations (42).

According to Appendix C, the continuous fraction (55) is always convergent. The WDE of γ_1 then reads

$$\gamma_1 \approx \langle \tilde{\tau}_{11} \rangle - \tfrac{1}{2} \langle \tilde{\tau}_{11}^2 \rangle + (\langle \tilde{\tau}_{12} \rangle - \langle \tilde{\tau}_{11} \tilde{\tau}_{12} \rangle)\, \tilde{u}_1 \tag{57}$$

The formulas (55) and (57) represent the main result of this section. In Appendix D we present applications of it.

To end this section, let us consider the special case of (24) when $\langle t_{12} \rangle = \langle t_{21} \rangle = 0$, $\langle t_{11} \rangle = -\langle t_{22} \rangle = \mu^2 \Delta$ $(\mu^2 \Delta \ll 1)$. Then

$$\langle \tilde{\tau} \rangle = \mu^2 \tilde{\Delta} \begin{pmatrix} x^* - x & 1 - x^2 \\ 1 - x^{*2} & x - x^* \end{pmatrix}, \qquad \tilde{\Delta} = \frac{\Delta}{1 - |x|^2} \tag{58}$$

Let us note that, in accordance with considerations in Appendix C, Δ does not influence the limiting behavior of u_n. For the small $\Delta \ll 1$ an expansion of u_n in powers of Δ is possible; the coefficients of the expansion could be calculated starting from (48). We have not, however, succeeded in constructing the expansion of u_n in powers of Δ^{-1} in the opposite limit $\Delta \gg 1$. As in this limit, according to (29), also the nondegenerate theory works, such an expansion could be easily calculated from (21), using (B5) and (B7).

It is interesting to note that, just as the nondegenerate theory from Section 2 is unsuitable to treat the degenerate problem, the degenerate theory described in this section is also inapplicable to the construction of the WDE of the LE for nondegenerate problem. This is, however, not a surprise, since for the last problem the variable u as defined in (30) is $1-O(\mu)$, and therefore the expansion of (34) with $\lambda_1-\lambda_2 \sim 1$ is no longer possible.

3.3. The Case $|\lambda_1| = |\lambda_2|$, $\lambda_1 \neq \lambda_2$

It is easy to show that any real 2×2 matrix with eigenvalues which satisfy condition (7) can be written in the form

$$\mathbf{T}^{(l)} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} + \begin{pmatrix} t_{11}^{(l)} & t_{12}^{(l)} \\ t_{21}^{(l)} & t_{22}^{(l)} \end{pmatrix} \tag{59}$$

with $t_{ij}^{(l)} \sim \mu$ being random and real. To calculate the LE, we use again the definition (30), (31), from which the WDE can be found as

$$\begin{aligned}\gamma_1 = i\varphi + \langle \tau_{11}\rangle\, e^{-i\varphi} - \tfrac{1}{2}\langle \tau_{11}^2\rangle\, e^{-2i\varphi} \\ + \langle \tau_{12}e^{-i\varphi} - \tau_{11}\tau_{12}e^{-2i\varphi}\rangle\, u_1 \\ - \tfrac{1}{2}\langle \tau_{12}^2\rangle\, e^{-2i\varphi}u_2\end{aligned} \tag{60}$$

In (60), the matrix τ is defined as in (35) and the quantities u_1, u_2 are defined according to (39) with $u^{(l)}$ now satisfying the recursive relations

$$u^{(l)} = \frac{e^{-i\varphi}u^{(l-1)} + \tau_{21}^{(l)} + \tau_{22}^{(l)}u^{(l-1)}}{e^{i\varphi} + {}_{11}^{(l)} + \tau_{12}^{(l)}u^{(l-1)}} \tag{61}$$

Following the same consideration as in Section 3.2, we conclude from (61) that the WDE possess no anomalies unless

$$\varphi = \pi\frac{r}{s} \tag{62}$$

In the last case the recurrence relation (61) gives $u_n \sim \mu^n$ for $n < s$, but $u_s \sim \mu^{s-2}$, and so an anomaly appears in the $(2s-2)$th order of the expansion. Therefore, to calculate the coefficients of the WDE for $n < 2s-2$, one can use the nondegenerate perturbation from Section 2. For the higher coefficients, however, the degenerate perturbation is necessary.

For *real* random matrices $\mathbf{T}$ the anomaly in the second order of expansion appears only for $\varphi = \pi/2$. To treat it, we can construct the new matrix

$\tilde{\mathbf{T}}^{(l)} = \mathbf{T}^{(2l)}\mathbf{T}^{(2l-1)}$, which has a degenerate spectrum, and use the formulas from Section 3.2. (The same treatment for $s > 2$ confirms our claim that in this case no anomaly arises in the leading term of the WDE).

4. DISCUSSION

We have presented a new method of constructing the nondegenerate weak disorder expansion of Lyapunov exponents of the products of random matrices. This method is easy to generalize to a form applicable also in the case when a degeneracy of two eigenvalues makes the standard nondegenerate treatment inapplicable. For the real degenerate matrices, the algorithm which gives the leading term of the expansion is described.

In Section 3.2 we have obtained a simple formula which gives the second-order term of the degenerate WDE of the larger LE in the form of a continuous fraction. For all problems we have chosen, it converges very well and the obtained results are in agreement with numerical simulations.

To calculate the leading term of the expansion of the LE, we have used the fact that all matrices $\mathbf{X}^{(n)}$ in (1) which can be transformed into each other using the similarity transformation $\mathbf{X} \rightarrow \mathbf{U}^{-1}\mathbf{X}\mathbf{U}$ for any $\mathbf{U}$ have the same Lyapunov exponents. It is therefore enough to treat only one matrix out of this "class" of X's for which the LE are easy to calculate. Let us note in this connection that the transformation (46) with

$$\mathbf{U} = \begin{pmatrix} \Theta^{-1/2} & 0 \\ 0 & \Theta^{1/2} \end{pmatrix} \tag{63}$$

is equivalent to the change of the definition (30) to

$$\gamma_1 = \lim_{n \to \infty} \frac{1}{N} \sum_{l=1}^{n} \log \frac{y_1^{(l)} + i\Theta y_2^{(l)}}{y_1^{(l-1)} + i\Theta y_2^{(l-1)}} \tag{30a}$$

Thus, the symmetry of the LE as reported in ref. 15 is given by the symmetry of the LE with respect to the transformation (46), (63) with Θ, $\Theta \neq 0, \infty$.

The form of our result differs considerably from that given in ref. 15; unfortunately, we have not succeeded in proving their equivalence. Maybe this is caused by the completely different strategy of the solutions [in ref. 15, for instance, the resulting formula should be invariant with respect to any similarity transformation (46)]. From the practical point of view, however, it seems that the continuous fraction (55) converges better than the rather complicated integral in ref. 15. A comparison of both formulas provides the interesting possibility to find the connection between the rather complicated integrals given in ref. 15 and the corresponding continuous fractions.

Using the similarity transformation, we transformed our problem (1) to an equivalent one in which the matrices $\mathbf{T}$ are no longer real, but have the form (6). It is interesting at this point to show the connection between our expansion as given in Section 3 and the transfer-matrix method of Pendry *et al.*[20,23] To do so, let us remark that, instead of $\mathbf{T}$, one can consider other matrices $\tilde{\mathbf{T}}^{(l)}$, $\tilde{\mathbf{X}}^{(l)}$ of higher dimensionality, which satisfy the same recurrence relation

$$\tilde{\mathbf{X}}^{(l+1)} = \tilde{\mathbf{T}}^{(l+1)}\tilde{\mathbf{X}}^{(l)} \tag{64}$$

The elements of $\tilde{\mathbf{X}}$, $\tilde{\mathbf{T}}$ can be constructed as

$$\begin{aligned} \tilde{T}_{ij}^{(l+1)} &= F_{ij}(\{T_{kl}^{(l+1)}\}) \\ \tilde{X}_{ij}^{(l+1)} &= F_{ij}(\{X_{kl}^{(l+1)}\}) \end{aligned} \tag{65}$$

with the functions F_{ij} independent of l chosen in such a way that (64) holds. It is possible to construct many such matrices. The simplest examples provide the matrices $\mathbf{T}^{[p]}$ discussed in Section 2. Other examples have been found, using the method of tensor products and the theory of representation, in refs. 20 and 24. Other matrices with infinite dimensionality can be constructed, choosing, for instance, the function F_{11}, and then looking for functions F_{ij} which conserve the validity of the recurrence relation (64).

Let us consider now the 2×2 matrices $\mathbf{T}$ of the form (6) and choose the functions $F_{11} = 1$, $F_{1i} = 0$ for all i and $F_{21} = T_{21}/T_{11}$. Then, using relation (64), we can step by step determine all other functions F, and thus construct the matrix $\tilde{\mathbf{T}}$ which has the form

$$\tilde{\mathbf{T}} = \begin{pmatrix} 1 & 0 & 0 & & \\ \dfrac{T_{21}}{T_{11}} & \mathscr{D} & \left(-\dfrac{T_{12}}{T_{11}}\right)\mathscr{D} & \left(\dfrac{T_{12}}{T_{11}}\right)^2 \mathscr{D} & \cdots \\ \left(\dfrac{T_{21}}{T_{11}}\right)^2 & \cdots & & & \\ \left(\dfrac{T_{21}}{T_{11}}\right)^3 & \cdots & & & \\ \vdots & \ddots & & & \end{pmatrix} \tag{66}$$

In (66), $\mathscr{D} = (T_{11}T_{22} - T_{21}T_{12})/T_{11}^2$. As $\tilde{\mathbf{X}}$ is of the same form as $\tilde{\mathbf{T}}$, we have $\tilde{\mathbf{X}}_{n1} = \{X_{21}/X_{11}\}^n$, which is nothing but our parameter u_n from Section 3. In full analogy with considerations of Pendry *et al.*, we conclude that

$$u_1 = \lim_{N \to \infty} \{\langle \tilde{\mathbf{T}} \rangle^N\}_{21} \tag{67}$$

As u_1 is finite, the matrix $\langle \tilde{\mathbf{T}} \rangle$ has the largest eigenvalue $\lambda_{\max} = 1$; it is therefore enough to find only the corresponding eigenvector $\mathbf{u}$. Evidently $\mathbf{u} = (1, u_1, u_2,...)$, and the system of equations for the components of the vector $\mathbf{u}$ is identical with that in (40).

In constructing the WDE, we restricted ourselves to the case of real matrices. It is clear from the construction, however, that the same treatment works also for complex matrices of the form (6). It could be interesting to generalize it also for arbitrary complex matrices, or, equivalently, to real matrices with three or more times degenerate eigenvalues. Unfortunately, we have not succeeded in doing this. The direct application of the above method leads to difficulties which seem to be insuperable without new ideas.

APPENDIX A

Let us consider the $M \times M$ matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ such that

$$\mathbf{C} = \mathbf{AB} \tag{A1}$$

with the corresponding ones $\mathbf{A}^{[p]}$, $\mathbf{B}^{[p]}$, $\mathbf{C}^{[p]}$ constructed in the way described in Section 2.1. Using the well-known relations for the determinant, one has

$$C_{\alpha\beta}^{[p]} = \sum_{\Omega} (-1)^{P(\Omega)} C_{i_1 j_{\Omega(1)}} \cdots C_{i_p j_{\Omega(p)}} \tag{A2}$$

where $\omega_\alpha = (i_1,..., i_p)$, $\omega_\beta = (j_1,..., j_p)$. Ω is a permutation $(1,..., p) \to (\Omega(1),..., \Omega(p))$ and $P(\Omega)$ is equal to the number of pair exchanges which determine Ω.

Using (A1), we get

$$C_{\alpha\beta}^{[p]} = \sum_{\Omega} (-1)^{P(\Omega)} \sum_{k_1 \cdots k_p}^{M} A_{i_1, k_1} \cdots A_{i_p k_p} B_{k_1 j_{\Omega(1)}} \cdots B_{k_p j_{\Omega(p)}} \tag{A3}$$

Thanks to the factor $(-1)^{P(\Omega)}$, only terms with all k's different from each other survive in (A3). Then we can first sum over all different ordered sets

$\omega_\delta = (k_1,..., k_p)$ and then over all permutations Π of integers in the given set:

$$C_{\alpha\beta}^{[p]} = \sum_\delta \sum_\Pi \sum_\Omega (-1)^{P(\Omega)} A_{i_1,k_{\Pi(1)}} \cdots A_{i_p k_{\Pi(p)}} \times B_{k_{\Pi(1)} j_{\Omega(1)}} \cdots B_{k_{\Pi(p)} j_{\Omega(p)}} \tag{A4}$$

Substituting $\Omega \to \Pi^{-1}\Omega$ $[P(\Omega) = P(\Omega) - P(\Pi)]$, we get

$$\begin{aligned} C_{\alpha\beta}^{[p]} &= \sum_\delta \sum_\Pi (-1)^{P(\Pi)} A_{i_1,k_{\Pi(1)}} \cdots A_{i_p k_{\Pi(p)}} \\ &\quad \times \sum_\Omega (-1)^{P(\Omega)} B_{k_1 j_{\Omega(1)}} \cdots B_{k_p j_{\Omega(p)}} \\ &= \sum_\delta A_{\alpha\delta}^{[p]} B_{\delta\beta}^{[p]} \end{aligned} \tag{A5}$$

Owing to (A5),

$$\mathbf{C}^{[p]} = \mathbf{A}^{[p]} \mathbf{B}^{[p]} \tag{A6}$$

which proves (12).

There are two well-known special cases of the formula (A6): for $p = M$, $\mathbf{A}^{[p]} = \det \mathbf{A}$, and (A6) gives the formula for determinants. The case $p = M - 1$ corresponds, after a simple transformation, to the relation for inverse matrices. For details see ref. 16.

APPENDIX B

Using the definitions of $t_{\alpha\beta}^{[p]}$, $z_\alpha^{[p]}$, one can easily construct their WDE (up to the second order in μ):

$$t_{11}^{[p]} \approx \mu \sum_{k \leqslant p} \frac{B_{kk}}{\lambda_k} + \frac{\mu^2}{2} \sum_{k,l \leqslant p} \frac{B_{kk} B_{ll} - B_{kl} B_{lk}}{\lambda_k \lambda_l} \tag{B1}$$

Let ω_α differ from 1 only by replacing $i \leftrightarrow r$, $1 \leqslant i \leqslant p$, $r > p$. Then $\sum_\alpha = \sum_{i \leqslant p} \sum_{r > p}$, and we have

$$t_{1\alpha}^{[p]} \approx \mu \frac{B_{ir}}{\lambda_i} + \mu^2 \sum_{j \leqslant p} \frac{B_{ir} B_{jj} - B_{jr} B_{ij}}{\lambda_i \lambda_j} \tag{B2}$$

$$t_{\alpha\alpha}^{[p]} \approx \frac{\lambda_r}{\lambda_i} \left(\mu \sum_{k \in \alpha} \frac{B_{kk}}{\lambda_k} + \frac{\mu^2}{2} \sum_{k,l \in \alpha} \frac{B_{kk} B_{ll} - B_{kl} B_{lk}}{\lambda_k \lambda_l} \right) \tag{B3}$$

$$t_{11}^{[p]} t_{1\alpha}^{[p]} \approx \mu^2 \sum_{j \leqslant p} \frac{B_{jj} B_{ir}}{\lambda_i \lambda_j} \tag{B4}$$

Supposing that $\overline{z_\beta^{[p]}} \sim \mu^2$, for all β, the leading term of $\overline{z_\alpha^{[p]}}$ reads

$$\overline{z_\alpha^{[p]}} \approx \frac{\langle t_{\alpha 1}^{[p]} \rangle - \langle t_{\alpha 1}^{[p]} t_{11}^{[p]} \rangle}{1 - \langle t_{\alpha\alpha}^{[p]} \rangle} \approx -\mu^2 \sum_{k \neq i} \frac{\langle B_{rk} B_{ki} \rangle}{\lambda_k(\lambda_i - \lambda_r)} \tag{B5}$$

Defining an another set $\omega_\beta = (1, 2,..., j-1, j+1,..., p, s)$, we have

$$t_{1\alpha}^{[p]} t_{1\beta}^{[p]} \approx \mu^2 \frac{B_{ir} B_{js}}{\lambda_i \lambda_j} \tag{B6}$$

$$\overline{z_\alpha^{[p]} z_\beta^{[p]}} \approx \frac{\langle t_{\alpha 1}^{[p]} t_{\beta 1}^{[p]} \rangle}{1 - \langle t_{\alpha\alpha}^{[p]} t_{\beta\beta}^{[p]} \rangle} \approx \mu^2 \frac{\langle B_{ri} B_{sj} \rangle}{\lambda_i \lambda_j - \lambda_r \lambda_s} \tag{B7}$$

Finally, for $\omega_\alpha = (1, 2,..., i-1, i+1,..., j-1, j+1, r, s)$, $\sum_\alpha = \frac{1}{4} \sum_{ij \leqslant p} \sum_{rs > p}$ we obtain

$$t_{1\alpha}^{[p]} \approx \mu^2 \frac{B_{ir} B_{js} - B_{is} B_{jr}}{\lambda_i \lambda_j} \tag{B8}$$

$$\overline{z_\alpha^{[p]}} \approx \mu^2 \frac{\langle B_{ri} B_{sj} - B_{si} B_{rj} \rangle}{\lambda_i \lambda_j - \lambda_r \lambda_s} \tag{B9}$$

On the basis of the presented formulas, the WDE of Γ_p up to the fourth order could be easily constructed. We do not present here its expression, since it has been presented elsewhere.[14,15]

APPENDIX C

Let us suppose that (52) has the solution

$$x = e^{i\varphi}, \qquad \varphi \text{ real} \tag{C1}$$

After some simple mathematical operations, we can transform (49) to the form

$$\lim_{\mu \to 0} \frac{1}{\mu^2} \{ \langle [(t_{11}^{(l)} - t_{22}^{(l)}) \sin \varphi - (t_{12}^{(l)} + t_{21}^{(l)}) \cos \varphi - (t_{12}^{(l)} - t_{21}^{(l)})]^2 \rangle \} = 0 \tag{C2}$$

As φ is real, (C2) could be satisfied only if for any l

$$(t_{11}^{(l)} - t_{22}^{(l)}) \sin \varphi - (t_{12}^{(l)} + t_{21}^{(l)}) \cos \varphi - (t_{12}^{(l)} - t_{21}^{(l)}) \equiv 0 \tag{C3}$$

Relation (C3) represents a very strong correlation among the elements of the matrix $\mathbf{T}^{(l)}$. If it holds, then, using the matrix

$$\mathbf{U} = \begin{pmatrix} \cos(\varphi/2) & -\sin(\varphi/2) \\ \sin(\varphi/2) & \cos(\varphi/2) \end{pmatrix} \tag{C4}$$

in transformation (46), we obtain another random matrix $\tilde{\mathbf{T}}$, which has $\tilde{t}_{12}^{(l)} \equiv 0$ for *all l*. As mentioned in Section 3.1, such matrices are not of interest here.

Supposing that (C3) does not hold, we conclude that (52), (53) has no solution of the form (C1).

The fact that the absolute values of all four solutions of (52) differ from 1 has an important consequence. As the coefficients of the polynomial in (52) are equal to

$$\lim_{n \to \infty} \frac{1}{n} \mathscr{C}_{ni} \tag{C5}$$

with $\mathscr{C}$ given in (42), we have at the same time proved that the five-diagonal system (43) only has solutions which grow or decrease exponentially with n. We conclude therefore that the general solution of (43) can be written in the form

$$u_n = A_1 e^{-\chi_1 n} + A_2 e^{-\chi_2 n}, \qquad n \to \infty \tag{C6}$$

with

$$\operatorname{Re} \chi_1 > \operatorname{Re} \chi_2 > 0 \tag{C7}$$

It seems at first sight that to obtain u_1, u_2 it is possible to bound an infinite system (43) by choosing n_0 large enough, supposing $u_n = 0$ for $n > n_0$ and solving the resulting system of linear equations. Unfortunately, the resulting finite system is numerically unstable. Indeed, in the numerical solution we iterate the recursive relation (43) in the opposite direction (from n_0 to 0). We obtain therefore only the first term in (C6), in which the constant A_1 is very sensitive to the fluctuations on the rhs of (43).

We can repeat all the above considerations for the system (48): Supposing that in the limit $n \to \infty$, $\tilde{u}_n$ behaves as $\exp(i\tilde{\varphi}n)$, we obtain

$$\lim_{\mu \to 0} \frac{1}{\mu^2} \{\langle [(\tilde{t}_{11} - \tilde{t}_{22}) \sin \tilde{\varphi} - (\tilde{t}_{12} + \tilde{t}_{21}) \cos \tilde{\varphi} - (\tilde{t}_{12} - \tilde{t}_{21})]^2 \rangle\} = 0 \tag{C8}$$

To derive (C8), we have used the relations

$$\begin{aligned} \langle (\tilde{t}_{11} - \tilde{t}_{22})^2 \rangle &= \langle (\tilde{t}_{12} + \tilde{t}_{21})^2 \rangle \\ \langle (\tilde{t}_{11} - \tilde{t}_{22})(\tilde{t}_{12} + \tilde{t}_{21}) \rangle &= 0 \end{aligned} \tag{C9}$$

which are consequences of (47).

Relation (C8) assures that $\tilde{\varphi}$ is complex, i.e., $|\tilde{u}_n| \to 0$ for large n.

Owing to the symmetry of the matrix $\tilde{\tau}$ and of the coefficients $\mathscr{C}$, the second limiting solution increases exponentially.

The existence of only one exponentially decreasing limiting solution of (48) guarantees the convergence of the continuous fraction (55).

APPENDIX D

1. Consider the degenerate 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & \mu^2 \Delta \\ -\mu^2 \Delta & 1 \end{pmatrix}, \qquad \mathbf{B}_i = \begin{pmatrix} 0 & \varepsilon_1 \\ \varepsilon_2 & 0 \end{pmatrix} \tag{D1}$$

with ε_1 and ε_2 random, $\langle \varepsilon_1^2 \rangle = \langle \varepsilon_2^2 \rangle = 1$. In this special case it is not necessary to transform the original matrix $\mathbf{T}$. Indeed, the matrix τ now has the form

$$\tau = \frac{i}{2} \begin{pmatrix} -2\mu^2 \Delta + \mu(\varepsilon_2 - \varepsilon_1) & \mu(\varepsilon_2 + \varepsilon_1) \\ -\mu(\varepsilon_2 + \varepsilon_1) & 2\mu^2 \Delta - \mu(\varepsilon_2 - \varepsilon_1) \end{pmatrix} \tag{D2}$$

and so we have $\mathscr{C}_{n,n-1} = \mathscr{C}_{n,n+1} = 0$; Eqs. (40) therefore given directly

$$(2n-1)\, u_{2n-2} + (12n - 8i\Delta)\, u_{2n} + (2n+1)\, u_{2n+2} = 0 \tag{D3}$$

The WDE of the LE reads

$$\gamma_1 = \mu^2[-i\Delta + \tfrac{1}{4}(1 + u_2)] \tag{D4}$$

Taking $u_n = u_n^{(0)} + \Delta u_n^{(1)} + \cdots$, one obtains from (D2) the systems of linear equations for the quantities $u_n^{(i)}$. To obtain the real part of the LE, we need only $u_2^{(0)}$:

$$u_2^{(0)} = \cfrac{-1}{12 - \cfrac{9}{24 - \cfrac{25}{36 - \cdots}}} \tag{D5}$$

which provides

$$\frac{\operatorname{Re} \gamma_1}{\mu^2} \approx 0.228473 + O(\Delta^2) \tag{D6}$$

To find the imaginary part of the LE, one first has to calculate from (D2) all the quantities $u_n^{(0)}$ and use them in the rhs of a linear system for the variables $u_n^{(1)}$. We have done this numerically using only ten equations (D3), and obtain

$$\frac{\operatorname{Im}\gamma_1}{\mu^2} \approx 1.015080\Delta + O(\Delta^3) \tag{D7}$$

The results (D6), (D7), from which one can calculate[19,21] the radius of localization and the density of states of electrons with energy $E = \mu^2 \Delta$ in the one-dimensional Anderson model, argee with results obtained previously.[15,17–22]

2. Consider the 2×2 matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{B}_i = \begin{pmatrix} a\varepsilon_1 & \varepsilon_2 \\ \varepsilon_3 & b\varepsilon_4 \end{pmatrix} \tag{D8}$$

Now Eqs. (52), (53) are equal. We have therefore $x = y$ and the matrix $\mathbf{U}$ has the simple form

$$\mathbf{U} = \begin{pmatrix} \Theta & 0 \\ 0 & \Theta^{-1} \end{pmatrix}, \qquad \Theta = \left(\frac{1+x}{1-x}\right)^{1/2} \tag{D9}$$

where Θ solves the biquadratic equation

$$\Theta^4 - (a^2 + b^2)\,\Theta^2 + 1 = 0 \tag{D10}$$

For $a = b = 1$ we obtain $\Theta = 1$; in this special case no transformation of the original matrix is necessary, and from (57) we have directly $\gamma_1 = 0$, as shown in ref. 15. One easily finds also that in (42) only $\mathscr{C}_{n,n}$ differs from zero. Therefore $u_n = 0$ for all n.

3. We have tested our method numerically for matrices

$$\mathbf{B} = \begin{pmatrix} a\varepsilon_1 & b\varepsilon_1 + \varepsilon_2 + c\varepsilon_4 \\ \varepsilon_3 & \varepsilon_4 \end{pmatrix} \tag{D11}$$

with the ε's random and independent with box distribution $P(\varepsilon) = 1$ for $-1/2 < \varepsilon < 1/2$, $P(\varepsilon) = 0$ otherwise, and for different values of constants a, b, c. For all systems we have studied, formula (55) converges quickly, and the results, independent of the choice of the root of Eq. (52), are in agreement with numerical simulations of products of 5 million matrices (1). As an example, we present in Table I results for some different choices of a, b, and c.

Table I

a	b	c	$\mathrm{Re}\,\gamma_1/(\mu^2\langle\varepsilon^2\rangle)$
5.0	0.0	0.0	-0.439
0.5	0.0	0.0	$+0.112$
5.0	1.0	0.0	-0.453
5.0	1.0	1.0	-0.403
1.0	3.0	3.0	$+0.853$
5.0	1.0	7.0	$+0.380$

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